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A VARIATIONAL METHOD OF DETERMINING THE EIGENFREQUENCIES OF A LIQUID IN A CHANNEL*

N.G. KUZNETSOV

Plane free oscillations of a liquid in channel of constant cross-section are considered. The spectral problem whose eigenfunctions are the stream functions is investigated for the first time in this setting. The eigenvalues of the spectral parameter occurring in the boundary condition on the free surface are determined by a variational method. The properties of the stream functions are identified, and these properties can be used to reconstruct the flow pattern. In particular, it is shown that the nodal lines on which the stream eigenfunction vanishes (compare with the membrane problem considered, e.g., in /1/) necessarily join the free surface with the bottom. It is proved that the first, and under certain conditions also the second, eigenvalue is simple. Upper bounds of these eigenvalues are obtained. One bound depends only on the geometrical characteristics of the flow region and the other only on the corresponding eigenfunction.

The linear eigenvalue problem of a liquid with a free surface has been traditionally analysed in terms of the velocity potential and formulated as a Steklov mixed problem. The latter has usually been solved by a variational method and a method based on potential theory. Both are applicable for a fairly wide class of regions and are convenient for numerical implementation. The research of the last decade is surveyed in /2/, which describes the results of non-Soviet authors, and in /3/, which primarily focuses on Soviet studies.

1. The spectral problem for the stream function. Assume that an ideal incompressible heavy liquid in equilibrium fills a channel of cross-section W. The simply connected region $W \subset R_-^2 = \{(x, y) : y < 0\}$ has a piecewise boundary without cusp points, and $\partial W = F \cup B$, $F \cap B = \varnothing$. Here the free surface of the liquid $F = (a_-, a_+)$ lies on the abscissa axis and the entire curve B (the channel bottom) lies in R_-^2 with the exception of its ends $(a_\pm, 0)$ which are the corner points of ∂W.

Plane free time-harmonic oscillations of a liquid in a channel are usually described by the spectral problem /2-6/

$$\nabla^2 u = 0 \text{ in } W,\ u_y - \nu u = 0 \text{ on } F,\quad \partial u/\partial n = 0 \text{ on } B \cap R_-^2 \tag{1.1}$$

which necessitates finding the eigenvalues of the parameter ν and the corresponding real eigenfunctions from the Sobolec space $H^1(W)$ that satisfy the condition $\int_F u\,dx = 0$. Here n is the outer normal to ∂W, νg (g is the free-fall acceleration) is the square of the free oscillation frequency, and the function u is the oscillation velocity potential up to a time-harmonic multiplier.

Problem (1.1) has a discrete spectrum $0 < \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_n \leqslant \ldots$ which can be determined by the variational method, and the functional to be minimized is given by

$$\int_W |\nabla u|^2\,dx\,dy \Big/ \int_F u^2\,dx$$

Let ν be an eigenvalue of problem (1.1) and u the corresponding eigenfunction. Denote by v the harmonic function in W which is the conjugate of u (the stream function). This function is defined, apart from a constant term, which is chosen so that $v = 0$ on B. This is feasible because $v = \text{const}$ on B due to the condition on B in (1.1) and the Cauchy-Riemann equations. If we replace u_y with $-v_x$ in the condition on F in (1.1) and then differentiate the resulting equality with respect to x and again apply the Cauchy-Riemann

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equations, we obtain the relationship

$$v_{xx} + \nu v_y = 0 \text{ on } F \tag{1.2}$$

Thus, ν is an eigenvalue and v is the corresponding eigenfunction in the problem

$$\nabla^2 v = 0 \text{ in } W,\ v_{xx} + \nu v_y = 0 \text{ on } F,\ v = 0 \text{ on } B \tag{1.3}$$

We can similarly show that any eigenvalue of problem (1.3) is an eigenvalue of problem (1.1).

The generalized solution of problem (1.3) is the function $v \in H_B^1(W) \cap H_0^1(F)$, that satisfies the integral identity

$$\int_F v_x \eta_x \, dx - \nu \int_W \nabla v \cdot \nabla \eta \, dx\, dy = 0 \tag{1.4}$$

for any function $\eta \in H_B^1(W) \cap H_0^1(F)$. Here $H_B^1(W)$ is the subspace in $H^1(W)$ that consists of functions that vanish on $B \cap R_-^2$ /7, Sect.7.1.5/ and $H_0^1(F)$ is the closure of the set of smooth functions with a compact support on F in the norm of $H^1(F)$.

By (1.2), problem (1.3) can be reduced to the operator spectral problem $L\varphi - \nu M\varphi = 0$ in the space $L_2(F)$. Here L is a positive definite operator with the domain $D_L = H^2(F) \cap H_0^1(F)$ that acts by the formula $L\varphi = -d^2\varphi/dx^2$, and the operator M is defined as follows. Its domain is $D_M = H_0^1(F)$; $\varphi \in D_M$ is continued harmonically to the region W (the harmonic continuation is also denoted by φ) so that $\varphi = 0$ on B. Then $(M\varphi)(x) = \varphi_y(x, 0)$.

The existence of an infinite sequence of eigenvalue $0 < \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_n \leqslant \ldots \nu_n \to \infty$ as $n \to \infty$, for the equation $L\varphi - \nu M\varphi = 0$ can be established by passing to problem (1.1). Indeed, the energy space of the operator L is $H_0^1(F)$ /8, Sect.9/. We also know /7, Ch.7/ that $H^{1/2}(F)$ is the energy space of the positive definite operator M. The required fact now follows /8, Sect.44/ from the compact embedding of $H_0^1(F)$ in $H^{1/2}(F)$.

By the variational eigenvalue method /8/, we have

$$\nu_n = \inf \{\langle L\varphi, \varphi\rangle / \langle M\varphi, \varphi\rangle : \varphi \in H_0^1(F), \\ \langle M\varphi, \varphi_k\rangle = 0,\ k = 1, \ldots, n-1\} \tag{1.5}$$

where φ_k is the eigenelement of the equation $L\varphi - \nu M\varphi = 0$ that corresponds to the eigenvalue ν_k, and $\langle \cdot, \cdot\rangle$ is the duality relation between $H^{-s}(F)$ and $H^s(F)$ extending the scalar product in $L_2(F)$. Here and henceforth, ν_1 is defined in the same way as ν_n without the last condition under inf.

By (1.4), Eq.(1.5) is equivalent to the following:

$$\nu_n = \inf \Big\{\int_F v_x^2 dx \Big/ \int_W |\nabla v|^2 dx\, dy : v \in H_B^1(W) \cap H_0^1(F), \\ \int_W \nabla v \cdot \nabla v_k \, dx\, dy = 0, \quad k = 1, \ldots, n-1\Big\} \tag{1.6}$$

where v_k is the eigenfunction of problem (1.3) corresponding to the eigenvalue ν_k.

2. ***Nodal lines of stream eigenfunctions.*** The nodal lines of stream eigenfunctions are the lines where $v_k = 0$. By the maximum principle for harmonic functions, it follows that the stream eigenfunction v_k does not have isolated roots in W. The region $V \subset W$ is called the nodal region of the function v_k if $v_k \neq 0$ in V and ∂V consists of a nodal line of the function v_k and possibly sections of the free surface F and arcs of the curve B (compare the corresponding definition for the membrane problem /1/).

The nodal line of the function v_k obviously is not closed and does not have both end points on B. Indeed, otherwise, by the uniqueness theorem for the Dirichlet problem, $v_k \equiv 0$ in the region enclosed by the nodal line or between B and the nodal line and thus $v_k \equiv 0$ in W.

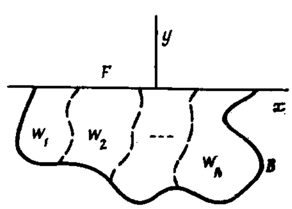

Fig.1

Lemma 1. Any nodal line of the stream eignefunction of a liquid has one end point on F and other end point on B and the nodal lines do not intersect (Fig.1).

Proof. We will show that the nodal line of the function v_k does not join two points on the free surface F. In fact, otherwise let

$$\varphi_1 = \begin{cases} v_k & \text{in } W' \\ 0 & \text{in } W \setminus W' \end{cases}$$

Here W' is the nodal region between F and the nodal line with both its end points on F. Introduce the functions $\varphi_i(x, y) = \varphi_1(x + t_i, y)$ $(i = 2, \ldots, k)$, where $t_i \neq t_j$ for $i \neq j$ and all t_i are such that a shift by t_i along the abscissa axis takes the region W' to a region contained in W. Continue the functions φ_i $(i = 2, \ldots, k)$ as zero so that they are defined everywhere in W. Since the functions φ_i $(i = 1, \ldots, k)$ are linearly independent, there exists a non-trivial linear combination $\Phi = a_1\varphi_1 + \ldots + a_k\varphi_k$ such that

$$\int_F \Phi\,(\partial v_j/\partial y)\,dx = 0, \quad j = 1, \ldots, k-1$$

which are equivalent to the equalities

$$\int_W \nabla\Phi\cdot\nabla v_j\,dx\,dy = 0, \quad j = 1, \ldots, k-1$$

The function Φ belongs to the class $H_B^1(W) \cap H_0^1(F)$ and satisfies the condition $\Phi_{xx} + \nu_k\Phi_y = 0$ on F. It therefore minimizes the functional (1.6). Hence we conclude that Φ is an eigenfunction and is therefore harmonic in W. Since Φ is identically zero in some subregion contained in W, then $\Phi \equiv 0$ in W, and we obtain a contradiction.

Theorem 1. The stream eigenfunction v_n corresponding to the eigenvalue ν_n changes its sign on F at most $n-1$ times. Its nodal lines partition W into at most n subregions.

Proof. By our lemma, the nodal lines of the function v_n partition W into a certain number of subregions N (Fig.1). Assume that $N > n$ and define the functions

$$\psi_i = \begin{cases} v_n & \text{in } W_i \\ 0 & \text{in } W\setminus W_i \end{cases}$$

which belong to the class $H_B^1(W) \cap H_0^1(F)$. These functions are linearly independent and a linear combination $\Psi = b_1\psi_1 + \ldots + b_n\psi_n$ exists that satisfies the orthogonality conditions $\int_W \nabla\Psi\cdot\nabla v_j dxdy = 0,\ j = 1, \ldots, n-1$ (compare with the proof of the lemma). Since $\Psi_{xx} + \nu_n\Psi_y = 0$ on F, then Ψ minimizes the functional (1.6), which shows that Ψ is harmonic in W. Since $N > n$, we have $\Psi \equiv 0$ in some subregion contained in W. We have obtained a contradiction. The theorem is proved.

Remark 1. The proof of Theorem 1 relies on the idea used in Courant's nodal line theorem for a vibrating membrane (see, e.g., /1/). It was also shown in /1/ that for the membrane we have the inequality $N_n < n$ for sufficiently large n, where N_n is the number of nodal regions of the n-th eigenfunction. The example of a rectangular channel for which

$$v_n = \sin\frac{\pi n x}{l}\,\mathrm{sh}\,\frac{\pi n(y+d)}{l}, \quad \nu_n = \frac{\pi n}{l}\,\mathrm{th}\,\frac{\pi n d}{l} \tag{2.1}$$

where $(0, l)$ is the free surface and $(-d, 0)$ is the channel wall, shows that the number of nodal regions of the function v_n in the oscillating liquid problem may equal n for all $n = 1, 2, \ldots$.

Corollary 1. The trace on F of the eigenfunction v_n corresponding to the eigenvalue ν_n has at most n extremum points. In particular, v_1 (we assume that it is positive in W) has precisely one maximum on F (Fig.2), which coincides with $\max\{v_1(x, y)\colon (x, y) \in \overline{W}\}$.

Proof. At the extremum point, the derivative $(\partial v_n/\partial x)(x, 0)$ changes its sign. Pass to the conjugate function u_n harmonic in W. By the Cauchy-Riemann equations and the condition on F from (1.1), $u_n(x, 0)$ also changes its sign at the extremum point of the function $v_n(x, 0)$. It has been shown /9/ that $u_n(u_1)$ changes its sign on F at most n times (precisely once). Therefore, $v_n(x, 0)$ has at most n extremum points ($v_1(x, 0)$ has precisely one maximum) on F (at least one maximum exists for $v_1(x, 0)$, because $v_1(a_\pm, 0) = 0$). The equality $\max\{v_1(x, 0)\colon x \in (a_-, a_+)\} = \max\{v_1(x, y)\colon (x, y) \in \overline{W}\}$ holds by the maximum principle for harmonic functions and the boundary conditions in (1.3).

Remark 2. By Corollary 1, the level lines of the function v_1 (the flowlines of the liquid wave motion for the principal eigenfrequency) have the form shown in Fig.3. The difference between the values of the function v_1 corresponding to adjacent lines is constant (compare with Fig.2).

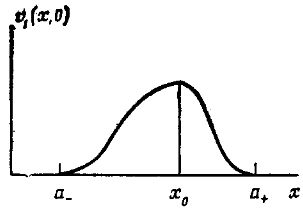

Fig.2

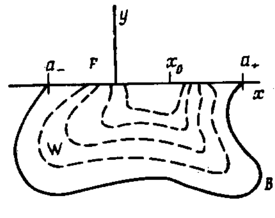

Fig.3

Corollary 2. The eigenvalue ν_1 is simple.

Proof. Assume that two eigenfunctions v' and v'' correspond to the eigenvalue ν_1. Then by Theorem 1 both functions may be regarded as positive. Denote by M' and M'' the maxima of the functions v' and v'' which are attained at the points $(x', 0)$ and $(x'', 0)$, respectively (see Corollary 1). If $x' \neq x''$, then the function $M''v' - M'v''$ changes its sign on F, which contradicts Theorem 1. If $x' = x''$, then the function $M''v' - M'v''$ either identically vanishes, which contradicts the linear dependence of v' and v'', or has three zeros on F and therefore no fewer than two extrema on F, which contradicts Corollary 1.

Corollary 3. If the trace $v_1(x, 0)$ does not have points of inflection on F, then v_1 is the only stream function which is sign-constant on W. In this case, the function v_2 has precisely one nodal line.

Proof. By assumption, the derivative $\partial^2 v_1/\partial x^2$ and therefore $\partial v_1/\partial y$ (see the boundary condition on F in (1.3)) are sign-constant on F. By formula (1.6), the eigenfunctions $v_2, \ldots, v_n, \ldots$ should satisfy the orthogonality condition

$$\int_W \nabla v_n \cdot \nabla v_1 \, dx \, dy = \int_F v_n (\partial v_1/\partial y) \, dx = 0, \quad n = 2, 3, \ldots$$

By the sign-constancy of the derivative $\partial v_1/\partial y$, the functions $v_2, \ldots, v_n, \ldots$ should change their sign. By Theorem 1, the function v_2 has at most one nodal line, and therefore it has precisely one nodal line.

Remark 3. If the eigenfunction corresponding to the eigenvalue ν_2 has a nodal line, then by Corollary 1 in each of the two nodal regions the flowlines have the form shown in Fig.3.

Corollary 4. If only stream eigenfunctions with nodal lines correspond to the eigenvalue ν_2, then ν_2 is simple.

Proof. Assume that two linearly independent eigenfunctions v' and v'' with nodal lines correspond to the eigenvalue ν_2. By Theorem 1, each function changes its sign F precisely once and can be chosen so that $\int_F v'v''dx = 0$. Therefore, the points where v' and v'' change their sign are distinct. Without loss of generality we may take $v'(x, 0) \gtrless 0$ for $x \gtrless x'$ and $v''(x, 0) \lesseqgtr 0$ for $x \gtrless x''$, where $x' < x''$.

Consider the expression $w(x, t) = (1 - t)v'(x, 0) + tv''(x, 0)$, where $t \in [0, 1]$. Clearly, for any $t \in [0, 1]$ and $x \in (x', x'')$, we have the inequality $w(x, t) > 0$. Define the sets

$$T' = \{t \in [0, 1]: w(x, t) < 0 \text{ for some } x < x'\},$$
$$T'' = \{t \in [0, 1]: w(x, t) < 0 \text{ for some } x > x''\}.$$

We see that $T' = [0, t')$ and $T'' = (t'', 1]$ for some t' and t''. If $t'' < t'$, then for $t \in (t'', t')$ the function $w(x, t)$ changes its sign on F no less than twice, which contradicts Theorem 1. If $t' \leqslant t''$, then for $t \in [t', t'']$ and all $x \in (a_-, a_+)$ we have the inequality $w(x, t) \geqslant 0$, which contradicts the condition.

Remark 4. The method of proving Corollary 4 was used in /9/ to prove that the eigenvalue ν_1 is simple. Moreover, propositions similar to Lemma 1 and Theorem 1 were proved in /9/ for the velocity potentials of plane modes of a liquid in a channel.

Now consider the existence of points of inflection of the trace on F of the eigenfunction v_1. We will need a lemma, which follows from the results of /10/.

Lemma 2. Let β be the angle between F and the one-sided tangent to B at the point $(a_-, 0)$; $\beta \in (0, \pi)$. Then for $\beta \neq \pi/2$ the asymptotic expansion of the eigenfunction in the neighbourhood of the point $(a_-, 0)$ has the form

$$v_n = C_1^{(n)} \{\rho^{\pi/\beta} \sin(\pi\theta/\beta) - \nu_n\beta(\pi+\beta)^{-1}\rho^{1+\pi/\beta}[\cos(1+\pi/\beta)\theta + \operatorname{ctg}\beta \sin(1+\pi/\beta)\theta]\} + C_2^{(n)}\rho^{2\pi/\beta}\sin(2\pi\theta/\beta) + v_n^* \tag{2.2}$$

Here v_n^* is a function of class $C^2(\overline{W})$ such that the relationship $v_n^* = o(\rho^{2+\delta})$ holds for any $\delta \in [0, 1)$; ρ, θ are polar coordinates with the origin $(a_-, 0)$ and the axis directed along $F, \theta \in (-\beta, 0)$.

For $\beta = \pi/2$, (2.2) is replaced with the asymptotic formula $v_n = C_1^{(n)}\rho\cos\theta + v_n^*$, where $v_n^* \in C^1(\overline{W})$ and the relationship $v_n^* = o(\rho^{1+\delta})$ holds for any $\delta \in [0, 1)$.

A similar proposition holds for the point $(a_+, 0)$.

Remark 5. In formula (2.2), at least one of the constants $C_k^{(n)}$ (for $k > 2$ then enter the expansion of the function v_n^*) should be non-zero, because otherwise by the "strong" zero theorem /11/ the function v_n vanishes identically in the neighbourhood of the point $(a_-, 0)$ which is impossible.

For a positive function v_1 in W we have the inequality $C_1^{(1)} < 0$. Indeed, if we assume that this is not so, then the inequality $v_1 > 0$ breaks down in the neighbourhood of the point $(a_-, 0)$.

Now from (2.2) we have

Corollary 5. If $\beta \neq \pi/2$, then $(\partial v_n/\partial x)(a_-, 0) = 0$, $n = 1, 2, \ldots$. A similar proposition holds for the point $(a_+, 0)$.

Theorem 2. Assume that at least one of the two angles between F and the one-sided tangents to B at the points $(a_\pm, 0)$ is not $\pi/2$. Then the function $v_1(x, 0)$ has at least one point of inflection.

Proof. Without loss of generality, we may assume that the angle with the apex at the point $(a_-, 0)$ is not $\pi/2$. By Corollary 5, we have the equality $(\partial v_1/\partial x)(a_-, 0) = 0$ and therefore near the point $(a_-, 0)$ the positive function $v_1(x, 0)$ is convex downward. Near the maximum point (see Corollary 1), this function is convex upward, which implies the existence of a point of inflection for y $v_1(x, 0)$.

Remark 6. Example (2.1) shows that the condition $\beta \neq \pi/2$ in Corollary 5 is essential; the function $v_1(x, 0)$ does not necessarily have a point of inflection if both angles between F and B are $\pi/2$.

3. *Bounds on the eigenvalues*. For any $x \in (a_-, a_+)$ let $d(x) = \min\{|y| : (x, y) \in B\}$, $d = \sup\{d(x) : x \in (a_-, a_+)\}$. Let $v \in H_B^1(W)$, and therefore the function $v(x, y)$ is absolutely continuous in y for almost all $x \in (a_-, a_+)$, so that

$$|v(x, 0)| = \Big|\int_{-d(x)}^{0} v_y(x, y)\,dy\Big| \leqslant \Big(d\int_{-d(x)}^{0} |v_y(x, y)|^2\,dy\Big)^{1/2}$$

Integrating over F, we obtain by (1.6)

$$\nu_1 \leqslant d \inf\Big\{\int_F v_x^2\,dx \Big/ \int_F v^2\,dx : v \in H_B^1(W) \cap H_0^1(F)\Big\} \tag{3.1}$$

Any function from $H_0^1(F)$ obviously can be continued in W so that its continuation belongs to $H_B^1(W)$. Therefore from (3.1) we have

$$\nu_1 \leqslant d \inf\Big\{\int_F v_x^2\,dx \Big/ \int_F v^2\,dx : v \in H_0^1(F)\Big\}$$

The last inf gives the first eigenvalue of the operator L, which equals $(\pi/l)^2$, where $l = a_+ - a_-$. We have thus proved the following theorem.

Theorem 3. $\nu_1 \leqslant d(\pi/l)^2$.

Remark 7. The inequality in Theorem 3 is exact in the following sense (isoperimetric). For a channel with a rectangular cross-section of width l and depth d, the first eigenvalue (see (2.1)) is equivalent to $d(\pi/l)^2$ when $d/l \to 0$. The ratio of the left- and right-hand sides of the inequality from Theorem 3 therefore goes to 1 for rectangular channels in which the ratio of the depth to the width tends to zero.

Remark 8. If the region W is contained in a rectangle having the same free surface, then

Theorem 3 is obtained from (2.1) and the following fact (see, e.g., /4, 6/). If two regions have the same free surface and one of the regions is contained in the other, then the first eigenvalue is greater for the larger region. At the same time, Theorem 3 is quite general, which is evident from examining the region shown in Fig.4.

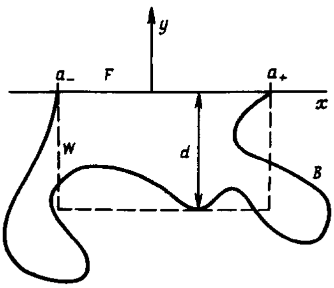

Fig.4

Remark 9. The method of deriving inequality (3.1) was applied in /12/ in connection with some auxiliary problem which has no physical meaning,

Theorem 4.

$$\frac{\pi}{4}\nu_1\int\limits_W v_1^2\,dx\,dy \leqslant \max\left\{\left|\frac{\partial v_1}{\partial x}(x,0)\right| : x\in[a_-,a_+]\right\}\cdot\int\limits_W v_1\,dx\,dy \tag{3.2}$$

Proof. The identity

$$\int\limits_W v^2\,dx\,dy = -\int\limits_W v\,[xv_x + yv_y]\,dx\,dy$$

is obtained directly by integration by parts. Here and henceforth, the subscript 1 is omitted for simplicity.

Define the curvilinear coordinate system (σ, ζ), where σ is the arc length along the level line of the function v and ζ is the length of the perpendicular to the level line (for details, see /13, p.413/). Let $W(v^*)$ be the subregion contained in W where $v > v^*$. Let

$$H_0(v^*) = \int\limits_{W(v^*)} v\,dx\,dy;\quad H_1(v^*) = -\int\limits_{W(v^*)} v\,[xv_x + yv_y]\,dx\,dy$$

By the shape of the level lines of the function v (Fig.3), these quantities may be expressed as follows:

$$H_0(v^*) = \int\limits_{v^*}^{v_m} v\,dv\int\limits_{B(v)} |\nabla v|^{-1}\,d\sigma$$

$$H_1(v^*) = \int\limits_{v^*}^{v_m} v\,dv\int\limits_{B(v)} (xn_x + yn_y)\,d\sigma;\quad B(v) = \partial W(v)\diagdown\bar F$$

Here v_m is the maximum value of the function v in $\bar W$, (n_x, n_y) is the outer normal to $\partial W(v)$. Clearly,

$$\int\limits_{B(v)} (xn_x + yn_y)\,d\sigma = 2A(v)$$

where $A(v)$ is the area of the region $W(v)$.

By the classical isoperimetric inequality, we have

$$-\frac{dH_1}{dv^*} = 2v^*A(v^*) \leqslant \frac{v^*}{2\pi}|\partial W(v^*)|^2 \leqslant \frac{2v^*}{\pi}|B(v^*)|^2 \tag{3.3}$$

($|\cdot|$ is the length of the line). At the same time, by the Cauchy-Bunyakovskii-Schwarz inequality, we may write

$$-dH_0/dv^* = v^* \int_{B(v^*)} |\nabla v|^{-1} d\sigma \geqslant v^* \Big[\int_{B(v^*)} d\sigma\Big]^2 \Big/ \int_{B(v^*)} |\nabla v| d\sigma =$$
$$v^* |B(v^*)|^2 \Big/ \Big[-\int_{B(v^*)} \partial v/\partial n \, d\sigma\Big]$$

By Green's formula and the condition on F from (1.3), the denominator of the last expression equals

$$\int_{a_-(v^*)}^{a_+(v^*)} v_y \, dx = -\nu_1^{-1} [v_x(x, 0)]_{x=a_-(v^*)}^{x=a_+(v^*)}$$

where $(a_\pm(v^*), 0)$ are the end points of the level line $B(v^*)$ located on F. By Corollary 1, the right-hand side of the last equality is positive and does not exceed

$$K = 2\nu_1^{-1} \max \{|v_x(x, 0)| : x \in [a_-, a_+]\}$$

which is finite by Lemma 2. Therefore

$$-dH_0/dv^* \geqslant v^* |B(v^*)|_l^2/K$$

Comparing this inequality with (3.3), we obtain

$$d[H_1 - 2\pi^{-1}KH_0]/dv^* \geqslant 0$$

Integration of this inequality gives

$$H_1(v_m) - 2\pi^{-1}KH_0(v_m) \geqslant H_1(0) - 2\pi^{-1}KH_0(0)$$

Hence, using the definition of H_0 and H_1, we obtain the required inequality, because $H_0(v_m) = H_1(v_m) = 0$.

Remark 10. The underlying idea of the proof of Theorem 4 was used in /14/ to obtain a lower bound on the first eigenvalue in the fixed membrane problem.

Corollary 6. Let v_2 be the stream eigenfunction corresponding to the eigenvalue ν_2. Then an inequality similar to (3.2) holds with ν_1 replaced by ν_2 and v_1 by $|v_2|$.

Proof. If the function v_2 is sign-constant (positive), then by Corollary 1 it has a unique maximum, which lies on F. Its level lines have the form shown in Fig.3. To obtain the required inequality, it suffices to repeat verbatim the proof of Theorem 3.

If the function v_2 has a nodal line (precisely one by Theorem 1), then its level lines in each nodal region have the form shown in Fig.3. Then by the proof of Theorem 4, for each nodal region W_i $(i = 1, 2)$ we obtain an inequality of the required form with W_i replacing W. Summing all these inequalities, we conclude the proof of Corollary 6.

Remark 11. Using the inequality $\Sigma a_i^2 \leqslant (\Sigma a_i)^2$, where $a_i \geqslant 0$, and a technique applied to prove Corollary 6 when v_2 has a nodal line, we can generalize the inequality from /14/ in the following way. Let λ_n be an eigenvalue and w_n the corresponding eigenfunction of the free oscillation problem for a fixed membrane. Then for any $n = 1, 2, \ldots$, we have the inequality

$$\int_D w_n^2 \, dx\, dy \leqslant \lambda_n (4\pi)^{-1} \Big(\int_D |w_n| \, dx\, dy\Big)^2$$

Here D is the region occupied by the membrane in the position of equilibrium.

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DIFFRACTION OF A SHORT ACOUSTIC WAVE BY A SMOOTH BODY WITH A DISCONTINUITY IN THE RADIUS OF CURVATURE OF ITS SURFACE*

V.N. LIKHACHEV

The propagation of a short acoustic wave in an ideal fluid when the radius of curvature of the wave front is discontinuous is considered. Such a wave arises if a short acoustic wave with a continuous radius of curvature is reflected from a smooth body whose surface has a discontinuity in the radius of curvature. The size of the body and its radius of curvature are assumed to be much greater than the wavelength.

In the immediate proximity of a body, an incident wave is reflected as a locally plane wave according to the laws of geometrical acoustics. Further from the body, geometrical convergence or divergence of rays begins to have an effect, and this determines the wave dynamics. If one of the radii of curvature of the body has a discontinuity along a line, the radius of curvature of the wave front also has a discontinuity, which lies on rays that originate from the points of the radius-of-curvature discontinuity line on the body surface. The geometrical acoustics solution produces different values of the wave amplitude on different sides of these rays, i.e., it has a strong tangential discontinuity and is thus inapplicable in the neighbourhood of rays that correspond to the curvature discontinuity of the wave front; diffraction of the reflected wave is observed in this region. We will derive a solution that describes the reflected wave everywhere, including the diffraction zone. The solution is obtained by matching asymptotic expansions, a method which has been previously applied to a number of other problems /1, 2/. The transverse profile of the wave is arbitrary and it is only required to satisfy the condition of zero perturbations on the leading characteristic.

Different wave-front geometries are possible. If the front is convex on both sies of the discontinuity, the diffraction zone goes to infinity. An interesting application of this problem is the design of a focusing reflector with a rounded edge. In this case, the intensity of the wave reflected from the concave reflector increases near the focus, while the intensity of the wave reflected from the convex edge decrease. The diffraction zone where these two geometrical acoustics solutions are matched may play an important role in flow calculations in the focal zone, because the opening angle of the focused wave decreases as we approach the focus while the diffraction zone increases. Our solution makes it

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